COMMUTING CONTRACTIVE IDEMPOTENTS IN MEASURE ALGEBRAS

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In honour of Tony's contributions to, and leadership in, the international abstract harmonic analysis community, the Canadian mathematical community, and my career.

Abstract. We determine when contractive idempotents in the measure algebra of a locally compact group commute. We consider a dynamical version of the same result. We also look at some properties of groups of measures whose identity is a contactive idempotent.

Let G be a locally compact group. When G is abelian, Cohen [1] characterised all of the idempotents in the measure algebra $\mathrm{M}(G)$. For non-abelian G, the idempotent probabilities were characterized by Kawada and Itô [3], while the contractive idempotents were characterized by Greenleaf [2]. We give an exact statement of their results in Theorem 0.1, below. For certain compact groups, the central idempotent measures were characterized by Rider [7], in a manner which is pleasingly reminiscent of Cohen's result on abelian groups. Rider points out a counterexample to his result when some assumptions are dropped. This has motivated our Example 1.3 (i), below.

Discussion of contactive idempotents has been conducted in the setting of locally compact quantum groups by Neufang, Salmi, Skalski and the present author [5].

Under certain assumptions, results of Stromberg [10] and Muhkerjea [4], show that convolution powers of a probability measure converge either to an idempotent, or to 0. See Theorem 2.1 and Remark 2.5, below. We study limits of convolution powers of products of contractive idempotents whose supports generate a compact subgroup.

We close with a study of certain groups of measures identified by Greenleaf [2] and Stokke [9] whose identities are contractive idempotents.

0.1. **Notation and background.** We shall always let G denote a locally compact group with measure algebra $\mathrm{M}(G)$. We let $\mathcal{K}(G)$ denote the collection of all compact subgroups of G. For K in $\mathcal{K}(G)$ we let m_K denote the normalised Haar measure on K as an element of $\mathrm{M}(G)$. We shall identify the group algebra $\mathrm{L}^1(K)$ as a subalgebra of $\mathrm{M}(G)$ via the identification $f \mapsto fm_K$, i.e. for $u \in \mathcal{C}_0(G)$ we

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.
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2010 *Mathematics Subject Classification.* Primary 43A05; Secondary 43A77, 43A40.
Key words and phrases. Measure algebra, idempotent, groups of measures.

define

$$\int_G u\, d(fm_K) = \int_K u(k)f(k)dk$$

where $dk = dm_K(k)$. We let for K in $\mathcal{K}(G)$, $\widehat{K}_1$ denote the space of multiplicative characters on K. Hence $\widehat{K}_1$ is the dual group of $K/[K,K]$, where $[K,K]$ is the closed commutator subgroup. If K is ableian we will write $\widehat{K}$ for $\widehat{K}_1$.

Let us recall what is known about contractive idempotents.

Theorem 0.1. **(i)** (Kawada and Itô [3]) *If μ in $\mathrm{M}(G)$ is a probability with $\mu * \mu = \mu$, then there is K in $\mathcal{K}(G)$ with $\mu = m_K$.*

(ii) (Greenleaf [2]) *If μ in $\mathrm{M}(G)$ is a non-zero and contractive, $\|\mu\| \leq 1$, and $\mu * \mu = \mu$, then there is K in $\mathcal{K}(G)$ and ρ in $\widehat{K}_1$ for which $\mu = \rho m_K$.*

Observe that all measures above are self-adjoint:

$$\int_G u\, d(\rho m_K)^* = \int_K u(k^{-1})\overline{\rho(k)}\, dk = \int_K u(k)\rho(k)\, dk = \int_G u\, d(\rho m_K)$$

thanks to unimodularity of the compact group K.

1. Main Result

In order to proceed, let us consider some conditions under which products of groups are groups.

Lemma 1.1. *Let $K_1, K_2 \in \mathcal{K}(G)$. Then the folowing are equivalent*

(i) $K_1K_2 = \{k_1k_2 : k_1 \in K_1, k_2 \in K_2\} \in \mathcal{K}(G)$,

(ii) K_1K_2 *is closed under inversion, and*

(ii) $K_1K_2 = K_2K_1$.

Proof. Note first that K_1K_2 is always a compact subset of G which contains the identity e. If (i) holds, then (ii) holds. We have that $(K_1K_2)^{-1} = K_2K_1$, which immediately shows the equivalence of (ii) and (iii). Finally if (iii) holds then it is clear that K_1K_2 is closed under multiplication. Thus, since (iii) implies (ii), we see that K_1K_2 is closed under multiplication and inversion, hence we obtain (i). $\square$

We observe that $K_1K_2 \in \mathcal{K}(G)$ in the following situations:

(i) $K_1 \subset K_2$, and

(ii) $K_1 \subset N_G(K_2) = \{s \in G : sK_2s^{-1} = K_2\}$.

If $K_1 \cap K_2 = \{e\}$ and $K_1K_2 \in \mathcal{K}(G)$, then (K_1, K_2) is referred to as a *matched pair* [12], and K_1K_2 is a *Zappa-Szép product* [13, 11]. Indeed, we note that the representation k_1k_2 of an element of K_1K_2 is unique for if $k_1k_2 = k_1'k_2'$, then $(k_1')^{-1}k_1 = k_2'k_2^{-1} = e$. Since, in general, we will not assume that $K_1 \cap K_2 = \{e\}$, nor even that this intersection is normal in K_1K_2, when the latter is a group, our situation appears to generalize that of a matched pair.

Is there a "nice" characterization of when $K_1K_2 \in \mathcal{K}(G)$?

To proceed we shall use a non-normal form of the Weyl integration formula. If H is a locally compact group and $L \in \mathcal{K}(H)$, then any continuous multiplicative function $\delta : L \to \mathbb{R}^{>0}$ is trivial. Thus the modular function Δ of H satisfies

$\Delta|_L = 1$, which is the modular function of L. Hence the left homogeneous space H/L admits a left H-invariant Haar measue $m_{H/L}$. We have for u in $\mathcal{C}_c(H)$ that

$$\int_H u(h)\,dh = \int_{H/L}\int_L u(hl)\,dl\,d(hL) \tag{1.1}$$

where $d(hL) = dm_{H/L}(hL)$.

Theorem 1.2. *Let $K_1, K_2 \in \mathcal{K}(G)$, $\rho_1 \in \widehat{(K_1)}_1$ and $\rho_2 \in \widehat{(K_2)}_1$. Then $\rho_1 m_{K_1}$ and $\rho_2 m_{K_2}$ commute if and only if one of the following cases holds for $K = K_1 \cap K_2$:*

(i) *$\rho_1|_K \neq \rho_2|_K$, in which case $(\rho_1 m_{K_1}) * (\rho_2 m_{K_2}) = 0$; or*

(ii) *$\rho_1|_K = \rho_2|_K$, $K_1K_2 \in \mathcal{K}(G)$ and the function*

$$\rho : K_1K_2 \to \mathbb{C} \text{ given by } \rho(k_1k_2) = \rho_1(k_1)\rho_2(k_2) \text{ for } k_1 \text{ in } K_1 \text{ and } k_2 \text{ in } K_2$$

defines a character; in which case $(\rho_1 m_{K_1}) * (\rho_2 m_{K_2}) = \rho m_{K_1K_2}$.

In particular, the idempotent probabilities m_{K_1} and m_{K_2} commute if and only if $K_1K_2 \in \mathcal{K}(G)$, and we have $m_{K_1} * m_{K_2} = m_{K_1K_2}$, in this case.

Proof. We let $\nu = (\rho_1 m_{K_1}) * (\rho_2 m_{K_2})$. Notice that

$$\rho_1 m_{K_1} \text{ and } \rho_2 m_{K_2} \text{ commute if and only if } \nu^* = \nu. \tag{1.2}$$

For u in $\mathcal{C}_0(G)$ we have

$$\begin{aligned} \int_G u\,d\nu &= \int_{K_1}\int_{K_2} u(k_1k_2)\rho_1(k_1)\rho(k_2)\,dk_1\,dk_2 \\ &= \int_{K_1/K}\int_K\int_{K_2} u(k_1kk_2)\rho_1(k_1k)\rho(k_2)\,dk_2\,dk\,d(k_1K) \\ &= \int_{K_1/K}\int_{K_2} u(k_1k_2)\int_K \rho_1(k_1k)\rho_2(k^{-1}k_2)\,dk\,dk_2\,d(k_1K) \qquad (1.3) \\ &= \int_{K_1/K}\int_{K_2}\left[\int_K \rho_1(k)\overline{\rho_2(k)}\,dk\right] u(k_1k_2)\rho_1(k_1)\rho_2(k_2)\,dk_2\,d(k_1K). \end{aligned}$$

The orthogonality of characters entails that the quantity $\int_K \rho_1(k)\overline{\rho_2(k)}\,dk$ is either 1 or 0, depending on whether $\rho_1|_K = \rho_2|_K$ or not. In the latter case, we see that $\nu = 0$, and hence $(\rho_2 m_{K_2}) * (\rho_1 m_{K_1}) = \nu^* = 0 = \nu$, and we see that condition (i) holds.

Hence for the remainder of the proof, let us suppose that $\rho_1|_K = \rho_2|_K$. Then the function $\rho : K_1K_2 \to \mathbb{T}$ given as in (ii) is well-defined. Indeed, if $k_1k_2 = k_1'k_2'$, then $(k_1')^{-1}k_1 = k_2'k_2^{-1} \in K$, and our assumption allows us to apply ρ_1 to the left, and ρ_2 to the right, to gain the same result. Furthermore, $(k_1, k_2) \mapsto \rho_1(k_1)\rho_2(k_2) = \rho(k_1k_2) : K_1 \times K_2 \to \mathbb{T}$ is continuous and hence factors continuously through the topological quotient space K_1K_2 of $K_1 \times K_2$.

We now wish to show that $\operatorname{supp}\nu = K_1K_2$. The inclusion $\operatorname{supp}\nu \subseteq K_1K_2$ is standard. Conversely, if k_1^o in K_1, k_2^o in K_2 and $\varepsilon > 0$ are given and let $u, v \in \mathcal{C}_0(G)$ satisfy

$$u \geq 0 \text{ and } u(k_1^o k_2^o) > \varepsilon > 0; \text{ and } v|_{K_1K_2} = \bar{\rho}.$$

Then we may find open U_1 containing k_1^o and open U_2 containing k_2^o so that $U_1\times U_2\subseteq\{(k_1,k_2)\in K_1\times K_2 : u(k_1k_2)>\varepsilon\}$, and our assumptions entail that

$$\begin{aligned}\int_G uv\,d\nu &= \int_{K_1}\int_{K_2} u(k_1k_2)\,dk_1\,dk_2\\ &\geq \int_{U_1}\int_{U_2} u(k_1k_2)\,dk_1\,dk_2 \geq m_{K_1}(U_1)m_{K_2}(U_2)\varepsilon > 0.\end{aligned}$$

Hence $K_1K_2\subseteq \operatorname{supp}\nu$. Notice that if it were the case that $\nu=0$, this would contradict our present calculation of $\operatorname{supp}\nu$, and hence the assumption that $\rho_1|_K=\rho_2|_K$. Thus $\nu=0$ only when $\rho_1|_K\neq\rho_2|_K$, showing that (i) fully characterizes this situation. We observe that

$$K_1K_2=\operatorname{supp}\nu^*=(\operatorname{supp}\nu)^{-1}=K_2K_1. \tag{1.4}$$

Let us now assume that $\rho_1m_{K_1}$ and $\rho_2m_{K_2}$ commute. Then, by (1.2), $\nu=\nu^*$ and hence by (1.4) and Lemma 1.1, we have that $K_1K_2\in\mathcal{K}(G)$. To complete the calculation we observe the following isomorphism of left K_1-spaces, generalizing the second isomorphism theorem of groups:

$$K_1K_2/K_2\cong K_1/K,\ kK_2\mapsto kK. \tag{1.5}$$

Hence for $u\in\mathcal{C}(K_1K_2)$ which is constant of left cosets of K_2 we have $\int_{K_1K_2/K_2}u(k)\,d(kK_2)=\int_{K_1/K}u(k_1)\,d(k_1K)$, for the unique choices of left-invariant probability measures on the homogeneous spaces. We thus find that

$$\begin{aligned}\int_G u\,d\nu &= \int_{K_1/K}\int_{K_2} u(k_1k_2)\rho(k_1k_2)\,dk_2\,d(k_1K)\\ &= \int_{K_1K_2/K_2}\int_{K_2} u(k_1k_2)\rho(k_1k_2)\,dk_2\,d(kK_2)\\ &= \int_{K_1K_2} u(k)\rho(k)\,dk=\int_G u\,d(\rho m_{K_1K_2})\end{aligned} \tag{1.6}$$

so $\nu=\rho m_{K_1K_2}$. Since $\nu*\nu=\nu$, as $\rho_1m_{K_1}$ and $\rho_2m_{K_2}$ commute, and $m_{K_1K_2}$ is the normalized Haar measure of a compact subgroup, it follows that $(\rho m_{K_1K_2})*(\rho m_{K_1K_2})=(\rho*\rho)m_{K_1K_2}$, whence $\rho=\rho*\rho$. We could appeal immediately to Theorem 0.1 (ii), to see that since $\|\nu\|\leq\|\rho_1m_{K_1}\|\|\rho_2m_{K_2}\|=1$, that $\rho\in\widehat{(K_1K_2)}_1$. However, let us give a direct verification, using only the present tools. We may interchange the roles of K_1 and K_2 above, and define $\tilde\rho:K_2K_1\to\mathbb{T}$ by $\tilde\rho(k_2k_1)=\rho_2(k_2)\rho_1(k_1)$, which, like ρ, is well-defined and continuous. We also see, by the computation (1.6), that $\nu=\tilde\rho m_{K_2K_1}=\tilde\rho m_{K_1K_2}$. Hence $\tilde\rho=\rho$ on K_1K_2. But it then follows that ρ is a homomorphism: if $k=k_1k_2$, $l=l_1l_2$, $k_1,l_1\in K_1$, $k_2,l_2\in K_2$, we have $k_2l_1=l_1'k_2'$ for some l_1' in K_1 and k_2' in K_2 and hence

$$\begin{aligned}\rho(k_1k_2l_1l_2)&=\rho_1(k_1l_1')\rho_2(k_2'l_2)=\rho_1(k_1)\rho(l_1'k_2')\rho_2(l_2)\\ &=\rho_1(k_1)\tilde\rho(k_2l_1)\rho_2(l_2)=\rho_1(k_1)\rho_2(k_2)\rho_1(l_1)\rho_2(l_2)=\rho(k_1k_2)\rho(l_1l_2).\end{aligned}$$

Conversely, if the conditions of (ii) are assumed, then computations (1.3) and (1.6) show that $(\rho_1m_{K_1})*(\rho_2m_{K_2})=\rho m_{K_1K_2}$ and show the same with the roles of $\rho_1m_{K_1}$ and $\rho_2m_{K_2}$, reversed. $\square$

Example 1.3. **(i)** Let $G = K \rtimes A$ where A is a compact group acting as continuous automorphisms on the group K, so we obtain group law $(k,\alpha)(k',\beta) = (k\alpha(k'),\alpha\beta)$. We identify K and A with their cannonical copies in G and suppose there is ρ in $\widehat{K}_1$ for which $\rho\circ\alpha \neq \rho$ for some α in A, and hence for α on an open subset of A. (A specific example would be to take $K = \mathbb{T}$, $A = \{\mathrm{id},\sigma\}$ where $\sigma(t) = t^{-1}$, and $\rho(t) = t^n$ where $n \in \mathbb{Z}\setminus\{0\}$.) Then for $u \in \mathcal{C}(G)$ we obtain for ρ as above

$$\int_G u\, d[(\rho m_K) * m_A] = \int_K \int_A u(k,\alpha)\rho(k)\, d\alpha\, dk$$

while, since the modular function on the compact group A qua automorphisms on K is 1, we have

$$\begin{aligned}\int_G u\, d[m_A * (\rho m_K)] &= \int_A \int_K u(\alpha(k),\alpha)\rho(k)\, dk\, d\alpha \\ &= \int_A \int_K u(k,\alpha)\rho\circ\alpha^{-1}(k)\, dk\, d\alpha.\end{aligned}$$

Thus ρm_K and m_A do not commute. The only assumption missing from Theorem 1.2 is that $(k,\alpha)\mapsto\rho(k)$ is not a character on G.

(ii) Let $n \geq 5$ and S_n the symetric group on a set of n elements, let S_{n-1} denote the stabiliser subgroup of any fixed element, and C the cyclic subgroup generated by any full n-cycle. Then $S_n = S_{n-1}C$, as may be easily checked, and $\{S_{n-1}, C\}$ is a "non-trivial" matched pair in the sense that neither subgroup is normal in G.

We note that the only non-trivial co-abelian normal subgroup of S_n is $A_n = \ker\mathrm{sgn}$, as A_n is simple and of index 2; hence $\widehat{(S_n)}_1 = \{1, \mathrm{sgn}\}$. Hence if ρ_2 in $\widehat{C}\setminus\{1\}$ satisfies $\rho_2 \neq \mathrm{sgn}|_C$, then for any ρ_1 in $\widehat{(S_{n-1})}_1$, it follows from Theorem 1.2 that $(\rho_1 m_{S_{n-1}}) * (\rho_2 m_C) \neq (\rho_2 m_C) * (\rho_1 m_{S_{n-1}})$.

2. Dynamical considerations

If S is a subset of G, let $\langle S\rangle$ denote the smallest closed subgroup containing S.

Theorem 2.1. (Stromberg [10]) *If μ is a probability in $\mathrm{M}(G)$, for which $K = \langle \mathrm{supp}\,\mu\rangle \in \mathcal{K}(G)$, then the weak* limit, $\lim_{n\to\infty}\mu^{*n}$, exists if and only if $\mathrm{supp}\,\mu$ is contained in no coset of a closed proper normal subgroup of K. Moroever, this limit equals the Haar measure m_K.*

We observew that $\mathrm{supp}\,\mu^* = (\mathrm{supp}\,\mu)^{-1}$, and hence in the assumptions above we have $\lim_{n\to\infty}(\mu^*)^{*n} = m_K$ too.

Since $\mathrm{supp}(m_K * m_L) = KL$, as was checked in the proof of Theorem 1.2, it follows that for K, L in $\mathcal{K}(G)$ for which $\langle KL\rangle$ is compact, we have $\lim_{n\to\infty}(m_K * m_L)^{*n} = m_{\langle KL\rangle} = \lim_{n\to\infty}(m_L * m_K)^{*n}$. For example, in $S = \mathrm{SU}(2)$, any two distinct (maximal) tori T_1 and T_2 generate S, as the only subgroups of S with non-trivial connected components are tori, or S, itself. Hence $m_S = \lim_{n\to\infty}(m_{T_1} * m_{T_2})^{*n}$.

Futhermore, we can deduce from the observation above that m_L and m_K commute if and only if $KL = \langle KL\rangle$, giving the special case of Theorem 1.2.

Motivated by the above considerations, we consider the following dynamical result.

Theorem 2.2. *Let $K_j \in \mathcal{K}(G)$ and $\rho_j \in \widehat{(K_j)}_1$ for $j = 1, \ldots, m$ for which $L = \langle K_1 \ldots K_m \rangle \in \mathcal{K}(G)$. Then the weak* limit*

$$\lim_{n\to\infty} [(\rho_1 m_{K_1}) * \cdots * (\rho_m m_{K_m})]^{*n}$$

always exists. It is ρm_L, provided there is a ρ in $\widehat{L}_1$ for which $\rho|_{K_j} = \rho_j$ for each j, and 0 otherwise.

Proof. We let $\nu = (\rho_1 m_{K_1}) * \cdots * (\rho_m m_{K_m})$. Then each ν^{*n}, being a product of contractive elements, satisfies $\|\nu^{*n}\| \leq 1$. The Peter-Weyl theorem tells us that the algebra $\mathrm{Trig}(L)$ consiting of matrix coefficients of finite-dimensional unitary representations, is uniformly dense in in $\mathcal{C}(L)$. Hence, since $\operatorname{supp} \nu \subseteq L$ and $\|\nu\| \leq 1$, hence $\|\nu^{*n}\| \leq 1$ for each n, it suffices to determine, for any finite dimensional unitary representation unitary $\pi : L \to \mathrm{U}(d)$, the nature of the limit

$$\lim_{n\to\infty} \pi(\nu^{*n}) = \lim_{n\to\infty} \int_L \pi(l)\, d\nu^{*n}(l) \text{ in } M_d(\mathbb{C}). \tag{2.1}$$

It is well-known, and simple to compute that each

$$\pi(\nu^{*n}) = \pi(\nu)^n = [\pi(\rho_1 m_{K_1}) \ldots \pi(\rho_1 m_{K_1})]^n.$$

For each $j = 1, \ldots, m$ the Schur orthogonality relations tell us that

$$\pi(\rho_j m_{K_j}) = \int_{K_j} \rho_j(k)\pi(k)\, dk = p_j$$

where p_j is the orthogonal projection onto the space of vectors ξ for which $\pi(k)\xi = \overline{\rho_j(k)}\xi$ for each k in K_j. Hence it follows that

$$\pi(\nu) = p_1 \ldots p_m \text{ and } \pi(\nu^{*n}) = (p_1 \ldots p_n)^n.$$

Since each p_j is contractive, the eigenvalues of $\pi(\nu)$ are of modulus not exceeding one. Furthermore, if $\|\pi(\nu)\xi\|_2 = \|\xi\|_2$ (Hilbertian norm), then we find that

$$\|\xi\|_2 = \|p_1 \ldots p_m \xi\|_2 \leq \|p_2 \ldots p_m \xi\|_2 \leq \cdots \leq \|p_m \xi\|_2 \leq \|\xi\|_2$$

so equality holds at each place. But we see then that ξ is in the range of p_m, hence of p_{j-1} if it is in the range of p_j, and thus in the mutual range R_π of each of $p_1, \ldots, p_m$. If we consider the Jordan form of $\pi(\nu) = p_1 \ldots p_m$, we see that $\lim_{n\to\infty} \pi(\nu)^n = q$, where q is the necessarily contractive, hence orthogonal, range projection onto R_π. But then for ξ in R_π and k_j in K_j, $j = 1, \ldots, m$, we have

$$\pi(k_1 \ldots k_n)\xi = \pi(k_1) \ldots \pi(k_m)\xi = \overline{\rho_1(k_1)} \ldots \overline{\rho_n(k_n)}\xi.$$

If we have $\xi \neq 0$, then $\mathbb{C}\xi$ is $\pi(K_1 \ldots K_m)$-invariant, hence π-invariant as $L = \langle K_1 \ldots K_m \rangle$. Moreover, there is, then, ρ in $\widehat{L}_1$ for which $\pi(l)\xi = \rho(l)\xi$, and it follows that $\rho|_{K_j} = \rho_j$. Notice that this ρ is determined independently of the choice of ξ, and hence even the choice of π. In particular, if no such ρ exists, i.e. for every finite dimensional unitary representation $R_\pi = \{0\}$, then we have

$\lim_{n\to\infty}\nu^{*n}=0$, in the weak* sense. When this ρ does exists, we see for u in $\mathcal{C}_0(G)$ that each $\int_G u\,d(\nu^{*n})$ is given by

$$\begin{aligned}
&\int_{K_1}\cdots\int_{K_m}\cdots\cdots\int_{K_1}\cdots\int_{K_m} u(k_{11}\dots k_{1m}\dots k_{n1}\dots k_{nm})\\
&\qquad\qquad \rho_1(k_{11})\dots\rho_m(k_{1m})\dots\rho_1(k_{n1})\dots\rho_m(k_{nm})\,dk_{nm}\dots dk_{n1}\dots dk_{1m}\dots dk_{11}\\
&\qquad =\int_{K_1}\cdots\int_{K_m}\cdots\cdots\int_{K_1}\cdots\int_{K_m} u(k_{11}\dots k_{nm})\rho(k_{11}\dots k_{nm})\,dk_{nm}\dots dk_{11}\\
&\qquad =\int_G u\rho\,d([m_{K_1}*\cdots*m_{K_m}]^{*n}). \qquad (2.2)
\end{aligned}$$

It is easy to verify, as in the proof of Theorem 1.2, that $\sup(m_{K_1}\dots m_{K_m})=K_1\dots K_m$. Hence by Theorem 2.1 we have obtain weak* limit

$$\lim_{n\to\infty}\nu^{*n}=\rho m_L$$

as desired. □

In fact, the above result generalizes the necessity direction of Theorem 1.2.

Corollary 2.3. *Let K_j and ρ_j, $j=1,\dots,m$, be as in Theorem 2.2, above, and $L=K_1\dots K_m$. If $\nu=(\rho_1 m_{K_1})*\cdots*(\rho_m m_{K_m})$ is idempotent then either $\nu=0$, or $L=\langle L\rangle\in\mathcal{K}(G)$ and there is ρ in $\widehat{L}_1$ with $\rho|_{K_j}=\rho_j$ for each j.*

Proof. Suppose $\nu\neq 0$. By a similar method as in the proof of Theorem 1.2, we see that $\operatorname{supp}\nu=L$. Moreover, if ν is idempotent, then $\lim_{n\to\infty}\nu^{*n}=\nu$. Hence we obtain that $L=\langle L\rangle$, and there exists a multiplicative character ρ on L, as promised, thanks to Theorem 2.2. □

Though Corollary 2.3 generalizes the necessity direction of Theorem 1.2, the proof of the earlier result is more self-contained, not relying on Stromberg's result. Furthermore, the sufficiency direction of Theorem 1.2 cannot be generalized so easily, even with probability idempotent measures.

Example 2.4. The special orthogonal group $S=\mathrm{SO}(3)$ admits the well-known Euler angle decomposition: $S=T_1T_2T_1$ where

$$T_1=\left\{k_1(t)=\begin{bmatrix}1&0&0\\0&\cos t&-\sin t\\0&\sin t&\cos t\end{bmatrix}:0\le t\le 2\pi\right\}\text{ and}$$
$$T_2=\left\{k_2(t)=\begin{bmatrix}\cos t&-\sin t&0\\ \sin t&\cos t&0\\0&0&1\end{bmatrix}:0\le t\le 2\pi\right\}.$$

We note that multiplication $T_1\times(T_2/\{I,k_2(\pi)\})\times T_1\to S$ is a diffeomorphism. For u in $\mathcal{C}(S)$ we have

$$\int_{T_1}\int_{T_2}\int_{T_1}u\,d(m_{T_1}*m_{T_2}*m_{T_1})=\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}u(k_1(t_1)k_2(t_2)k_1(t_3))\,\frac{dt_3\,dt_2\,dt_1}{8\pi^3}$$

whereas the Haar measure m_S gives integral

$$\int_S u\,dm_S = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} u(k_1(t_1)k_2(t_2)k_1(t_3))\sin t_2\,\frac{dt_3\,dt_2\,dt_1}{8\pi^2}.$$

Hence considering T_1-spherical functions, i.e. u in $\mathcal{C}(T_1\backslash S/T_1)$, we see that

$$m_{T_1T_2T_1} = m_S \neq m_{T_1} * m_{T_2} * m_{T_1}.$$

Remark 2.5. We note the following result, shown (implicitly) by Muhkerjea [4, Theo. 2]. *If μ is a probability in $\mathrm{M}(G)$, for which $\langle \operatorname{supp}\mu\rangle \notin \mathcal{K}(G)$, then the weak* limit satisfies* $\lim_{n\to\infty}\mu^{*n} = 0$.

Hence if $K_1,\dots,K_m$ in $\mathcal{K}(G)$ have $\langle K_1\dots K_m\rangle \notin \mathcal{K}(G)$, we see that $\lim_{n\to\infty}(m_{K_1}*\dots*m_{K_m})^{*n} = 0$, which is rather antithetical to having $m_{K_1}*\dots*m_{K_m}$ be an idempotent.

As a simple example, consider the any two non-trivial finite subgroups K and L of discrete groups Γ and Λ, and consider each as a subgroup of the free product $\Gamma * \Lambda$. For a Lie theoretic example, consider the Iwasawa decomposition KAN of $S = \mathrm{SL}_2(\mathbb{R})$. Compute that if $a \in A\setminus\{I\}$, then $aKa^{-1} \neq K$. Since K is maximal compact, we see that $\langle KaKa^{-1}\rangle \notin \mathcal{K}(S)$.

It is the case that if for $K_1,\dots,K_m$ in $\mathcal{K}(G)$ we have $H = \langle K_1\dots K_m\rangle \notin \mathcal{K}(G)$, then for ρ_j in $\widehat{(K_j)}_1$, $j = 1,\dots,m$, the weak* limit satisfies

$$\lim_{n\to\infty}[(\rho_1 m_{K_1}) * \dots * (\rho_m m_{K_m})]^{*n} = 0. \tag{2.3}$$

In the case that $\rho_i|_{K_i\cap K_j} \neq \rho_j|_{K_i\cap K_j}$ for some $i \neq j$, we have $(\rho_1 m_{K_1}) * \dots * (\rho_m m_{K_m}) = 0$, as may be computed, by a straightforward adaptation of (1.3). If there is a continuous multiplicative character $\rho : H \to \mathbb{T}$ such that $\rho|_{K_j} = \rho_j$ for each j, then the computation (2.2), and Mukherjea's theorem give the result. In presence or absense of these assumptions, (2.3) follows from a result which should appear in work of Neufang, Salmi, Skalski and the author, in progress. In fact, the same result implies Theorem 2.2. However, the proof given in the present note uses simpler methods.

3. On groups of measures

Geenleaf's motivation for studying idempotent measures was their use in the study of contractive homomorphisms $\mathrm{L}^1(H) \to \mathrm{M}(G)$. In doing so, he required a description of certain groups of measures, given in Theorem 3.1, below. We are interested in determining how these groups interact under convolution product with each other. Stokke [9] conducted a study of Greenleaf's groups, and also devised a more general class of groups; see (3.1). We show that the latter class is indeed more general.

Let for any subgroup H of G

$$N_G(H) = \{g \in G : gHg^{-1} = H\} \text{ and } Z_G(H) = \{g \in G : gh = hg \text{ for all } h \text{ in } H\}$$

denote its normalizer and centralizer, respectively. Notice that for another subgroup L, we have $L \subseteq Z_G(H)$ if and only if $H \subseteq Z_G(L)$. Notice too that for

the topological closure $\overline{H}$, we have $N_G(\overline{H}) = N_G(H)$ and $Z_G(\overline{H}) = Z_G(H)$, and hence these subgroups are closed.

Given K in $\mathcal{K}(G)$ and ρ in $\widehat{K}_1$ we let

$$N_{K,\rho} = N_G(K) \cap N_G(\ker\rho)$$

and then let $q : N_{K,\rho} \to N_{K,\rho}/\ker\rho$ be the quotient map. We let

$$G_{K,\rho} = q^{-1}(Z_{N_{K,\rho}/\ker\rho}(K/\ker\rho)).$$

Hence g in $G_{K,\rho}$ normalizes both K and $\ker\rho$, and commutes with elements of K modulo $\ker\rho$. We then consider, in $\mathrm{M}(G)$, the subgroup

$$\Gamma_{\rho m_K} = \{z\delta_g * (\rho m_K) : z \in \mathbb{T} \text{ and } g \in G_{K,\rho}\}$$

We remark that $G_{K,\rho} = \{g \in G : \delta_g*(\rho m_K) = (\rho m_K)*\delta_g\}$, and $\Gamma_{\rho m_K}$ is topological group with the weak*-topology on $\mathrm{M}(G)$ and multiplication $z\delta_g * (\rho m_K) * z'\delta_{g'} * (\rho m_K) = zz'\delta_{gg'} * (\rho m_K)$.

Theorem 3.1. **(i)** (Greenleaf [2]) *Any closed group of contractive measures has identity of the form ρm_K of Theorem 0.1 (ii), and is a subgroup of $\Gamma_{\rho m_K}$.*

(ii) (Stokke [9], after [2]) *The map*

$$(z, g) \mapsto z\delta_g * (\rho m_K) : \mathbb{T} \times G_{K,\rho} \to \Gamma_{\rho m_K}$$

is continuous and open and with compact kernel $\{(\rho(k), k) : k \in K\} \cong K$.

Remark 3.2. We give a mild simplification of Stokke's argument, which will help us, below.

(i) *Let*

$$\Omega_{K,\rho} = (\mathbb{T} \times G_{K,\rho})/\{(\rho(k), k) : k \in K\}.$$

Then the one point compactification $\Omega_{K,\rho} \sqcup \{\infty\}$ (respectively, topological coproduct, if $G_{K,\rho}$ is compact) is homeomorphic to $\Gamma_{\rho m_K} \cup \{0\}$.

Indeed, consider the semigroup homomorphism on $(\mathbb{T} \times G_{K,\rho}) \sqcup \{\infty\}$ given by $(z, g) \mapsto z\delta_g*(\rho m_K)$, $\infty \mapsto 0$, which has kernel $\{(\rho(k), k) : k \in K\}$ at the identity — a fact which we shall take for granted, thanks to arguments in [9, 2]. It suffices to verify that this semigroup homorphism is continuous and that $\Gamma_{\rho m_K} \cup \{0\}$ is weak*-compact. Let (z_i, g_i) be a net in $\mathbb{T} \times G_{K,\rho}$ such that $z_i\delta_{g_i} * (\rho m_K) \to \mu$ in i. If (g_i) is unbounded in $G_{K,\rho}$, we may pass to subnet and assume $g_i \to \infty$. But then for u in $\mathcal{C}_0(G)$, $(u(g_i\cdot))$ converges to zero uniformly on compact sets, thanks to uniform continuity of u. It follows that $\mu = 0$. Otherwise (g_i) is bounded in $G_{K,\rho}$, and by passing to subnet, we may assume that $(z_i, g_i) \to (z, g)$ in $\mathbb{T} \times G_{K,\rho}$. But then for u in $\mathcal{C}_0(G)$, $(u(g_i\cdot))$ converges to $u(g\cdot)$ uniformly on compact sets, and it follows that $\mu = z\delta_g * (\rho m_K)$. Notice that any limit point of a net in $\Gamma_{\rho m_K}$ is in $\Gamma_{\rho m_K} \cup \{0\}$, so the latter set is weak*-closed, hence weak*-compact as it is a subset of the weak*-compact unit ball of $\mathrm{M}(G)$.

(ii) *If H is any closed subgroup of $G_{K,\rho}$, then*

$$\big((\mathbb{T} \times H)/\{(\rho(k), k) : k \in K \cap H\}\big) \sqcup \{\infty\}$$

is homoemorphic to $\{z\delta_g * (\rho m_K) : z \in \mathbb{T} \text{ and } g \in H\} \cup \{0\}$. Moroever, the latter set is weak-compact.* These facts are immediate from (i), above.

For a set Σ of contractive measures, let $[\Sigma]$ denote the smallest weak*-closed semigroup containing Σ.

Proposition 3.3. *Suppose K_1, K_2, ρ_1 and ρ_2 satisfy the contditions of Theorem 1.2 (ii), and let ρ be as given there. Then*

$$\left[\Gamma_{\rho_1 m_{K_1}} * \Gamma_{\rho_2 m_{K_2}}\right] \cap \Gamma_{\rho m_{K_1K_2}} = \{z\delta_g * (\rho m_{K_1K_2}) : z \in \mathbb{T}, g \in \langle H_1 H_2\rangle\}$$

where $H_1 = G_{K_1,\rho_1} \cap G_{K_1K_2,\rho}$ and $H_2 = G_{K_2,\rho_2} \cap G_{K_1K_2,\rho}$.

Proof. Let us record some observations about contractive idempotents. First we have that $\mathrm{supp}(\rho m_K) = K$. If g in G and z in $\mathbb{T}$ are is such that $z\delta_g * (\rho m_K) = \rho' m_{K'}$, then $gK = \mathrm{supp}(z\delta_g * (\rho m_K)) = \mathrm{supp}(\rho' m_{K'}) = K'$, so $K = K'$ and $g \in K$.

To see the inclusion of the first set into the second, let $g_1 \in G_{K_1,\rho_1}$, $g_2 \in G_{K_2,\rho_2}$. Then we compute

$$\delta_{g_1} * (\rho_1 m_{K_1}) * \delta_{g_2} * (\rho_2 m_{K_2}) = \delta_{g_1} * (\rho m_{K_1K_2}) * \delta_{g_2} = \delta_{g_1g_2} * \delta_{g_2^{-1}} * (\rho m_{K_1K_2}) * \delta_{g_2}$$

where $\delta_{g_2^{-1}} * (\rho m_{K_1K_2}) * \delta_{g_2}$ is a contractive idempotent. If we assume that there is g in $G_{K_1K_2,\rho}$ and z in $\mathbb{T}$ for which

$$\delta_{g_1g_2} * \delta_{g_2^{-1}} * (\rho m_{K_1K_2}) * \delta_{g_2} = z\delta_g * (\rho m_{K_1K_2}).$$

then it follows from the argument in the paragraph above that $g^{-1}g_1g_2 \in K_1K_2$. Hence $g^{-1}g_1 \in K_1K_2 \subseteq G_{K_1K_2,\rho}$ so $g_1 \in H_1$. Also, as $g \in N_G(K_1K_2)$, we have $g_2 \in K_1K_2g \subseteq G_{K_1K_2,\rho}$, and we obtain that $g_2 \in H_2$. By Remark 3.2 (ii), any non-zero limit of products of elements of $\{z\delta_g * (\rho m_{K_1K_2}) : z \in \mathbb{T}, g \in \langle H_1H_2\rangle\}$ remains in that set.

To see the reverse inclusion, we let $g_1 \in H_1$ and $g_2 \in H_2$ and we observe that

$$\delta_{g_1g_2} * (\rho m_{K_1K_2}) = \delta_{g_1} * (\rho m_{K_1K_2}) * \delta_{g_2} = \delta_{g_1} * (\rho_1 m_{K_1}) * \delta_{g_2} * (\rho_2 m_{K_2}).$$

We use Remark 3.2 (i), to see that non zero limits of products of such elements remain in $\Gamma_{\rho m_{K_1K_2}}$.

Either argument above can be easily redone, multiplied by elements of $\mathbb{T}$. □

Example 3.4. **(i)** In the notation above, suppose that $G_{K_1,\rho_1} = G$. The happens, for example, if K_1 is in the centre of G. Indeed, then $\ker\rho_1$ is in the centre of G, and $K_1/\ker\rho_1$ is in the centre of $G/\ker\rho_1$. Then, in the assumption of Proposition 3.3, we have $G_{K_1,\rho_1} \cap G_{K_1K_2,\rho} = G_{K_1K_2,\rho}$ and hence

$$\left[\Gamma_{\rho_1 m_{K_1}} * \Gamma_{\rho_2 m_{K_2}}\right] \cap \Gamma_{\rho m_{K_1K_2}} = \Gamma_{\rho m_{K_1K_2}}.$$

(ii) In the notation above, we always have that $K_1 \subseteq G_{K_1,\rho_1}$ and $K_2 \subseteq G_{K_2,\rho_2}$. Hence if $G = K_1K_2$, then by Proposition 3.3, we have

$$\left[\Gamma_{\rho|_{K_1} m_{K_1}} * \Gamma_{\rho|_{K_2} m_{K_2}}\right] \cap \Gamma_{\rho m_G} = \Gamma_{\rho m_G}$$

for any $\rho \in \widehat{G}_1$. This works even for "non-trivial" matched pairs in the sense of Example 1.3 (ii).

(iii) Let T be any non-trivial compact abelian group, σ be given on $T \times T$ by $\sigma(t_1, t_2) = (t_2, t_1)$ and $G = (T \times T) \rtimes \{\mathrm{id}, \sigma\}$. Let $\rho_1, \rho_2 \in \widehat{T}$ (dual group of T) so

$\rho_1 \times \rho_2 \in \widehat{T \times T}$. Then $N_G(T \times \{e\}) = T \times T$ is abelian and hence it is easy to follow the definition to see $G_{T\times\{e\},\rho_1} = T \times T$. By symmetry, $G_{\{e\}\times T,\rho_2} = T \times T$, as well.

On the other hand $N_G(T \times T) = G$, and $\sigma(\ker \rho_1 \times \rho_2) = \ker \rho_1 \times \rho_2$, so $N_G(\ker \rho_1 \times \rho_2) = G$. Also

$$G/\ker \rho_1 \times \rho_2 = [(T \times T)/\ker \rho_1 \times \rho_2] \rtimes \{\mathrm{id}, \sigma\} \cong \rho_1 \times \rho_2(T \times T) \times \{\mathrm{id}, \sigma\}$$

is abelian, i.e. σ acts trivially on the image $\rho_1 \times \rho_2(T \times T) \cong (T \times T)/\ker \rho_1 \times \rho_2$. Hence $G_{T\times T,\rho_1\times\rho_2} = G$. Thus by Proposition 3.3 we have

$$\left[\Gamma_{\rho_1 m_{T\times\{e\}}} * \Gamma_{\rho_2 m_{\{e\}\times T}}\right] \cap \Gamma_{(\rho_1\times\rho_2)m_{T\times T}} \subsetneq \Gamma_{(\rho_1\times\rho_2)m_{T\times T}}.$$

We now consider some groups of measures considered in [9]. For K in $\mathcal{K}(G)$ and ρ in $\widehat{K}_1$ let

$$\mathcal{M}_{\rho m_K} = \{\nu \in \mathrm{M}(G) : \nu^* * \nu = \rho m_K = \nu * \nu^*\}. \tag{3.1}$$

Notice that if $\nu \in \mathcal{M}_{\rho m_K}$, then the operator $\xi \mapsto \nu * \xi$ on $\mathrm{L}^2(G)$ is a partial isometry with support and range projection $\xi \mapsto (\rho m_K) * \xi$. Since the injection $\nu \mapsto (\xi \mapsto \nu * \xi)$ from $\mathrm{M}(G)$ into bounded operators on $\mathrm{L}^2(G)$ is injective, it follows that for ν in $\mathcal{M}_{\rho m_K}$ that $\nu * (\rho m_K) = \nu = (\rho m_K) * \nu$. We call $\mathcal{M}_{\rho m_K}$ the *local unitary group* at ρm_K. It is clear that $\Gamma_{\rho m_K} \subseteq \mathcal{M}_{\rho m_K}$.

Our goal is to make a modest determination of the scope of $\mathcal{M}_{m_K}$ for an idempotent probability measure. We begin with an analogue of a well-known characterization of the stucture of the connected component of the invertible group of a Banach algebra. This lemma plays more of a role in motivating the methods below, than in producing a result we shall use directly.

Lemma 3.5. *Let H be a locally compact group. Then the connected component of the identity of $\mathcal{M}_{\delta_e}$ in $\mathrm{M}(H)$ is the group*

$$\mathcal{M}_{\delta_e,0} = \{\exp \lambda_1 \dots \exp \lambda_n : \lambda_1, \dots, \lambda_n \in \mathrm{M}(H)_{\mathrm{ska}},\, n \in \mathbb{N}\}$$

where $\mathrm{M}(H)_{\mathrm{ska}} = \{\lambda \in \mathrm{M}(H) : \lambda^* = -\lambda\}$, the real linear space of skew-adjoint measures.

Proof. There exists norm-open neighbourhoods B of 0 and U of δ_e, in $\mathrm{M}(H)$, on which $\exp : B \to U$ is a homeomorphism. There is a logarithm defined on a neighbourhood of δ_e, and analytic functional calculus shows these are mutually inverse. We may suppose that B is symmetric and closed under the adjoint.

If $\nu \in U \cap \mathcal{M}_{\delta_e}$, then there is some λ in B for which $\nu = \exp \lambda$, and we have $\exp(\lambda^*) = \exp(\lambda)^* = \nu^* = \nu^{-1} = \exp(-\lambda)$, and hence $\lambda^* = -\lambda$, by assumption on B. If $\nu = \exp \lambda_1 \dots \exp \lambda_n$, with $\lambda_1, \dots, \lambda_n \in \mathrm{M}(H)_{\mathrm{ska}}$ and ν' in $\mathcal{M}_{\delta_e}$ is so close to ν that $\nu^* * \nu' \in U$, then $\nu^* * \nu' = \exp \lambda_{n+1}$ for some λ_{n+1} in $\mathrm{M}(H)_{\mathrm{ska}}$. The subgroup of all such products is hence open in $\mathcal{M}_{\delta_e}$ and clearly connected, thus the connected component of δ_e. $\square$

We say that a locally compact group H is *Hermitian* if each element self-adjoint element of $\mathrm{L}^1(H)$ has real spectrum. See [6] for notes on the class of Hermitian groups.

Proposition 3.6. *Let $K \in \mathcal{K}(G)$.*

(i) *If $N_G(K) \supsetneq K$, then $\Gamma_{m_K} \subsetneq \mathcal{M}_{m_K}$.*

(ii) *If $N_G(K)/K$ is contains either a non-discrete closed abelian subgroup, or a closed non-Hermitian subgroup, then the connected compoent of the identity $\mathcal{M}_{m_K,0}$, is unbounded.*

Proof. We let $H = N_G(K)/K$. We notice, in passing, that $N_G(K) = G_{K,1}$. The map $\varphi : \mathrm{M}(N_G(K)/K) \to \mathrm{M}(G)$ given for u in $\mathcal{C}_0(G)$ by

$$\int_G u\, d\varphi(\nu) = \int_{N_G(K)/K} \int_K u(gk)\, dk\, dg = \int_{N_G(K)} u(g)\, dg.$$

Since arbitrary elements of $\mathcal{C}_0(N_G(K)/K)$ may be represented as $gK \mapsto \int_K u(gk)\, dk$, as above, we see that φ is injective, even isometric. In particular

$$\mathcal{M}_{m_K} = \varphi(\mathcal{M}_{\delta_{e_H}}) \text{ and } \Gamma_{m_K} = \varphi(\Gamma_{\delta_{e_H}}) = \mathbb{T}\varphi(\delta_H)$$

where $\delta_H = \{\delta_h : h \in H\}$.

(i) To see that the inclusion $\Gamma_{m_K} \subseteq \mathcal{M}_{m_K}$ is proper, it suffices to see that $\Gamma_{\delta_{e_H}}$, is a proper subgroup of $\mathcal{M}_{\delta_{e_H}}$. Since H contains at least two elements, the real dimension of $\mathrm{M}(H)_{\mathrm{ska}}$ is at least 2. Since exp is analytic and a homeomorphism on a neighbourhood $\widetilde{B}$ of 0 in $\mathrm{M}(H)_{\mathrm{ska}}$, $\mathcal{M}_{\delta_{e_H}}$ contains a manifold or real dimension at least 2. But since δ_H is norm discrete, we can pick $\widetilde{B}$ small enough so that $\exp(\widetilde{B}) \cap \Gamma_{\delta_{e_H}} \subset \mathbb{T}\delta_{e_H}$. Hence $\exp(\widetilde{B}) \not\subset \Gamma_{\delta_{e_H}}$.

(ii) If there exists $\nu = \nu^*$ in $\mathrm{M}(H)$ with non-real spectrum, then the one-parameter subgroup $\{\exp(it\nu)\}_{t\in\mathbb{R}}$ is unbounded and a subgroup of $\mathcal{M}_{\delta_{e_H}}$. The Wiener-Pitt phenomenon shows that if H contains a closed non-discrete abelian subgroup A, then such a ν exists. Indeed, if $\nu = \nu^*$ in $\mathrm{M}(A) \subseteq \mathrm{M}(H)$, then the Fourier-Steiltjes transform satisfies $\hat{\nu} = \widehat{\nu^*} = \bar{\hat{\nu}}$, and we appeal to Section 6.4 in [8]. If H contains a closed non-Hermitian subgroup, we can choose ν to be absolutely continuous with respect to Haar measure. □

It is not clear whether or not $\mathcal{M}_{m_K}$ is always locally compact with respect to the weak* topology.

Remark 3.7. **(i)** The proof of (i) above tells us that if $N_G(K)/K$ is infinite, then $\mathcal{M}_{m_K}$ contains manifolds of arbitrarily high dimension. Thus we see that $\mathcal{M}_{m_K}$ is not Lie, in this case.

(ii) If $N_G(K)$ is compact, and hence so too is $H = N_G(K)/K$ with dual object $\widehat{H}$, then $\mathcal{M}_{m_K} \cong \mathcal{M}_{\delta_{e_H}}$ is a subgroup of the product of unitary groups $\prod_{\pi\in\widehat{H}} \mathrm{U}(d_\pi)$, containing the dense restricted product subgroup, consisting of all elements which are I_{d_π} for all but finitely many indices π. Indeed, $\nu \mapsto (\pi(\nu))_{\pi\in\widehat{H}} : \mathrm{M}(H) \to \ell^\infty\text{-} \oplus_{\pi\in\widehat{H}} M_{d_\pi}(\mathbb{C})$ (notation as in (2.1)) injects $\mathcal{M}_{\delta_{e_H}}$ into the product group. Furthermore, consider u in $\prod_{\pi\in\widehat{H}} \mathrm{U}(d_\pi)$ where $u_\pi = I_{d_\pi}$ for all but $\pi_1, \ldots, \pi_n$ in $\widehat{H}$, and $u_{\pi_k} = [u_{ij,k}]$ in $\mathrm{U}(d_{\pi_k})$ for $k = 1, \ldots, n$. The element of

$\mathrm{M}(H)$ given by

$$\nu_u = \delta_e + \sum_{k=1}^{n} d_{\pi_k} \left(\sum_{i,j=1}^{d_{\pi_k}} u_{ij,k}\pi_{k,ij} - \sum_{j=1}^{d_{\pi_k}} \pi_{k,jj} \right) m_H$$

where each set $\{\pi_{k,ij}\}_{i,j=1}^{d_{\pi_k}}$ are matrix coefficients of π_k with respect to an orthonormal basis for the space on which it acts, satisfies, with respect to the same basis, $\pi(\nu_u) = u_\pi$. Notice that $\nu_u * \nu_{u'} = \nu_{uu'}$ and $\nu_u^* = \nu_{u^*}$.

Acknowledgement. The author was supported by NSERC Grant 312515-2010. The author is grateful to M. Neufang for pointing him towards the concept of a matched pair, and to M. Neufang, P. Salmi and A. Skalski for helping him discover a mistake in an earlier version of Theorem 1.2. The author is also grateful to R. Stokke, for stimulating discussion of, and pointing out some errors in, the last section.

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